Tutorial 3 (Limits & Continuity)

(You should practise writing proper steps.)

A) Limits

1. Evaluate the following limits:

(a) $\lim_{x\to 2}(-x^2+x-2)$ (b) $\lim_{x\to 4} 2(x-3)(x-5)$ (c) $\lim_{x\to 2}\frac{x^2-2}{x-1}$ (d) $\lim_{x\to -1}\frac{2x^2-x-3}{x+1}$

(e) $\lim_{x\to 3}\frac{x^2+x-12}{2x-6}$ (f) $\lim_{x\to -1}\frac{x^3+1}{x+1}$ [Do you know how to factorise x^3+1?]

(g) $\lim_{x\to 4}\frac{\sqrt{x}-1}{x-1}$ (h) $\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

(i) $\lim_{x\to 0}\frac{\sqrt{3+x}-\sqrt{3}}{x}$ [The expression $\sqrt{3+x}+\sqrt{3}$ may be useful.]

(j) $\lim_{x\to 0}\frac{\frac{1}{2+x}-\frac{1}{2}}{x}$ [Simplify the expression first.] (k) $\lim_{x\to 3}\left[\frac{x^2-9}{x-3}-\frac{x^2-1}{x+1}\right]$

(l) $\lim_{x\to 0}\frac{\tan x}{x}$ [Hint: $\tan x=\frac{\sin x}{x}$] (m) $\lim_{x\to 2}\frac{|x-1|}{x-1}$ (n) $\lim_{x\to 2.5}\lfloor x\rfloor$

2. (a) Given that $\lim_{x\to 0}\frac{x}{\sin x}=L$, what is the value of L?

(b) Evaluate the following limits:

(i) $\lim_{u\to 0}\frac{\sin u}{u}$ (ii) $\lim_{x\to 0}\frac{\sin 3x}{x}$ (iii) $\lim_{x\to 0}\frac{\sin x^3}{x^3}$

(iv) $\lim_{x\to 0}\frac{2x(x+1)}{\sin 3x}$ [The expression $\frac{3x}{\sin 3x}$ may be useful.] (v) $\lim_{x\to 0} x^3\sin\frac{1}{x^3}$

3. (a) Determine $\lim_{x\to 3^+}\lceil x\rceil$ and $\lim_{x\to 3^-}\lceil x\rceil$. Does $\lim_{x\to 3}\lceil x\rceil$ exist? Why?

(b) Determine $\lim_{x\to 2^+}\frac{|x-2|}{x-2}$ and $\lim_{x\to 2^-}\frac{|x-2|}{x-2}$. Does $\lim_{x\to 2}\frac{|x-2|}{x-2}$ exist? Why?

4. (a) Given that $1-\frac{x^2}{4}\le f(x)\le 1+\frac{x^2}{2}$ for all $x\ne 0$, find $\lim_{x\to 0} f(x)$.

[Which theorem do you use?]

(b) Prove that $\lim_{x\to 0} x\cos\frac{1}{x}=0$. [Try to sandwich $x\cos\frac{1}{x}$ between two appropriate expressions.]

(c) Given that $3x-5\le f(x)\le x^2-3x+4$ for $x\ge 0$, find $\lim_{x\to 3} f(x)$

(d) Find $\lim_{x\to 0} x^4\sin\frac{1}{x^3}$.

5. Determine whether the limit exists by considering the corresponding one-sided limits. Give the value of the limit if it exists.

(a) $\lim_{x \to 2} f(x)$, $f(x) = \begin{cases} \dfrac{x+2}{2}, & x < 2 \\ \dfrac{12-2x}{3}, & x \geq 2 \end{cases}$

(b) $\lim_{x \to 2} f(x)$, $f(x) = \begin{cases} 3x-2 & x < 2 \\ \dfrac{6}{x}+1 & x \geq 2 \end{cases}$

(c) $\lim_{x \to 2} f(x)$, $f(x) = \begin{cases} 3-x & , x < 2 \\ 2 & , x = 2 \\ \dfrac{x}{2} & , x > 2 \end{cases}$

For each of the above, is f continuous at 2?

6. Determine the value of each of the following limits if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 2} \dfrac{|x^2-4|}{x-2}$ *Hint*: Use $|x^2-4| = \begin{cases} -(\ ?\ ), & \text{if } x < 2 \\ x^2-4, & \text{if } x < 2 \end{cases}$ or try another way.

(b) $\lim_{x \to 2} \left\lfloor \dfrac{x+1}{2} \right\rfloor$ [What are the values of $\left\lfloor \dfrac{x+1}{2} \right\rfloor$ for values of x near 2, say for $1 < x < 3$?]

B) Continuity

1. Determine whether the following functions are continuous at $x = 3$.

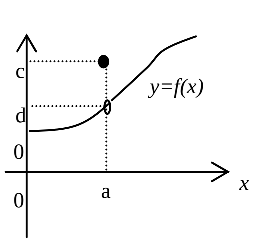
(a) $f(x) = \begin{cases} 2x^2 - 4 & (x > 3) \\ x + 11 & (x \leq 3) \end{cases}$

(b) $f(x) = \dfrac{2x}{3x^2 - 9x}$

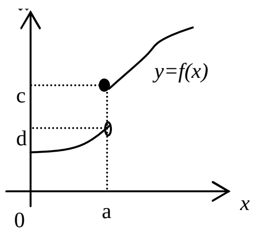
(c) $f(x) = \begin{cases} 3x-2 & x < 2 \\ \dfrac{6}{x}+1 & x \geq 2 \end{cases}$

2. For each of the functions graphed below, explain why the function is not continuous at $x = a$.

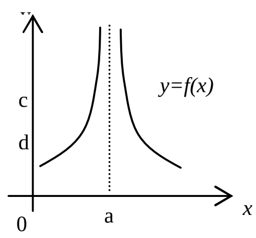
(a)



(b)



(c)



3. (a) If you look at the graph of $y = \tan x$, where are the points of discontinuity? Find the discontinuities of $f(x) = \tan(3x-2)$.

(b) Determine the discontinuities, if any, of the following function.

$$f(x) = \begin{cases} 2x+1, & x \le 0 \\ 1, & 0 < x \le 1 \\ x^2+1, & x > 1 \end{cases}$$

4. Evaluate each of the following limits by observing that each expression involved is a composition of continuous functions.

(a) $\lim_{x\to\pi} \cos(x+\sin x)$ (b) $\lim_{x\to 2} e^{x^2-1}$

5. At what points are the functions continuous?

(a) $f(x) = \dfrac{x+1}{x^2-3x+2}$ (b) $g(x) = \begin{cases} 3-x & if\ x \le 3 \\ 2x+1 & if\ x > 3 \end{cases}$

6. (a) Explain why $g(x) = \begin{cases} 3-x & if\ x \le 3 \\ 2x+1 & if\ x > 3 \end{cases}$ is a continuous function on the interval $(0,3)$ but not a continuous function on the interval $[0,3]$

(b) Explain why $g(x) = \dfrac{x}{x-2}$ is not a continuous function on the interval $[0,3]$

7. (a) State the intermediate value theorem (i.e. the full statement including the hypothesis and the conclusion).

(b) Show that there is a root of the equation $x - \cos x = 0$ in the interval $[0, \frac{\pi}{2}]$.

[Let $f(x) = x - \cos x$. Then apply Intermediate Value Theorem.]

8. (a) If $f(x) = x^3 - 8x + 10$, show that there is a value of c in the interval $(0,1)$ for which $f(c) = \pi$.

(b) Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 3 = 0$ in the interval $[1,2]$.

C) Limits involving infinity

1. Show that $\lim_{x\to 0^+}\left(\frac{1}{|x|}-\frac{1}{x}\right)$ exists but $\lim_{x\to 0^-}\left(\frac{1}{|x|}-\frac{1}{x}\right)$ does not.

   What can you conclude about $\lim_{x\to 0}\left(\frac{1}{|x|}-\frac{1}{x}\right)$? [Reminder: $|x| = \begin{cases} x & if\ x\geq 0 \\ -x & if\ x<0 \end{cases}$]

2. (a) Determine the vertical and/or horizontal asymptote(s) for the graph of each function defined as follows.

   [Consider $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, $\lim_{x\to a^-} f(x)$ and/or $\lim_{x\to a^+} f(x)$ for appropriate a.]

   (i) $f(x)=\dfrac{2x+1}{2x^2-5x+2}$

   (ii) $f(x)=\dfrac{3x^2-2x+4}{2x^2-5x+2}$ [Factoring $2x^2-5x+2$ may help.]

   (iii) $f(x)=\dfrac{x+3}{\sqrt{x^2+2x-8}}$

   (b) The graph of $f(x)=\dfrac{4x+8}{x^2-4}$ has a horizontal asymptote. Give the equation of this asymptote.

   Show that $x=-2$ is NOT a vertical asymptote for the graph of $f(x)=\dfrac{4x+8}{x^2-4}$.

3. For each of the following limits, determine if it exists. If it does not exist, could you write as $\lim_{x\to a} f(x)=\infty$ or $\lim_{x\to a} f(x)=-\infty$. Show steps to justify your answers.

(a) $\lim_{x\to+\infty}\dfrac{2x^3-4x}{5x^3+2}$ (b) $\lim_{x\to+\infty}\dfrac{5x^5-3}{3x^3-2}$ (c) $\lim_{x\to\infty}\left(\dfrac{8x^2+7}{2x^2}-\dfrac{9x^3+27}{3x^3-3}\right)$

(d) $\lim_{x\to\infty}\left(\sqrt{x(x+2)}-x\right)$ (e) $\lim_{x\to 2}\dfrac{2x+3}{3x^2-4x-4}$ (f) $\lim_{x\to\infty}\dfrac{2x+3}{3x^2-4x-4}$

(g) $\lim_{x\to 0}\dfrac{\cos x}{x^4}$ (h) $\lim_{x\to 0}\dfrac{\sin x}{x^4}$

(i) $\lim_{x->0}\dfrac{x^2+1}{x^4}$ (j) $\lim_{x->\infty}\dfrac{x^2+1}{x^4}$

(nby, Nov 2015)